CRITERIA FOR THE DETERMINATION OF THE "LIMITS" OF CONTINUOUS FLOW IN A FREELY EXPANDING JET

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UDC 533.70

We propose a criterion to evaluate the position of the "boundary" of a continuous medium in steady-state jets discharging into a vacuum. For a case of a spherical source the results obtained on the basis of this criterion are compared with the results obtained in a numerical solution of a model Boltzmann equation and through experimentation.

In calculating a freely expanding jet based on the model of a continuous medium it is necessary to determine when this model ceases to be valid and when we begin dealing with the flow of a rarefied gas. The answer to this question cannot be obtained with the Knudsen number, since in such a jet there is no control body nor a characteristic macroscopic dimension. Therefore, in determining the "boundary" of transition from a continuous medium to one that is expanded, we must resort to a kinetic equation or to certain criteria containing the local characteristics of the jet.

Until recently, the transition of a continuous medium into an expanding medium had been investigated only for symmetrical flows proceeding from spherical or cylindrical sonic sources of a gas. The basic research tool was the Bhatnagar–Gross–Krook (BGK) model [1] of the kinetic Boltzmann equation, written in an appropriate coordinate system. Relying on the BGK method in the formulation of reference [2], the authors of reference [3] found a sharp transition from continuous flow to the flow with a "frozen-in" temperature $T_\infty > 0$.

An experimental study is described in reference [4] for the process of the kinetic freezing-in of argon flow ($\varkappa = 5/3$) and the relationship between the limit Mach number and the characteristic Knudsen number, $Kn_0 = L_0/r^* = l_0$, has been determined, the latter calculated with respect to the mean free path in the decelerated gas and from the source radius r_*:

$$M_\infty = 1.37 Kn_0^{-0.4}. \quad (1)$$

The kinetic BGK model was also used in [5], but with consideration given to the terms omitted in [3]. Two conclusions were drawn here, and these differ from those drawn in [3]: 1) in the spherical case the transition to the frozen-in temperature T_∞ proceeds rather smoothly as $(T/T_\infty) - 1 \sim r^{-1}$; 2) in the cylindrical case the freezing-in of the temperature does not occur, regardless of the conditions.

In [6], in the case of spherical expansion, the problem reduces to a relaxation process with two translation temperatures along ($T_\parallel$) and across ($T_\perp$) the streamlines, corresponding to an ellipsoidal distribution function [7].

The process is a function of the Knudsen number and of the law governing the interaction between the molecules. The results of the calculations show that the region of transition is rather broad. Its "middle" is determined from the condition $T_\parallel - T_\perp = T$, which corresponds to the radius

$$r_1 = \frac{1}{5}(4.0395)^{\frac{3\beta}{3+4\beta}} \left[\frac{2}{3}\left(\frac{\pi\varkappa}{2}\right)^{\frac{1}{2}}\left(\frac{2}{\varkappa+1}\right)^{\frac{\varkappa+1}{2(\varkappa-1)}}\frac{1}{Kn_0}\right]^{\frac{3}{3+4\beta}}. \quad (2)$$

For argon ($\beta = 1/2$, $\varkappa = 5/3$) the finite Mach number is equal to

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Zhukovskii Central Aerohydrodynamic Institute, Moscow. Translated from Inzhenerno-Fizicheskii Zhurnal, Vol. 16, No. 1, pp. 9-14, January, 1969. Original article submitted March 18, 1968.

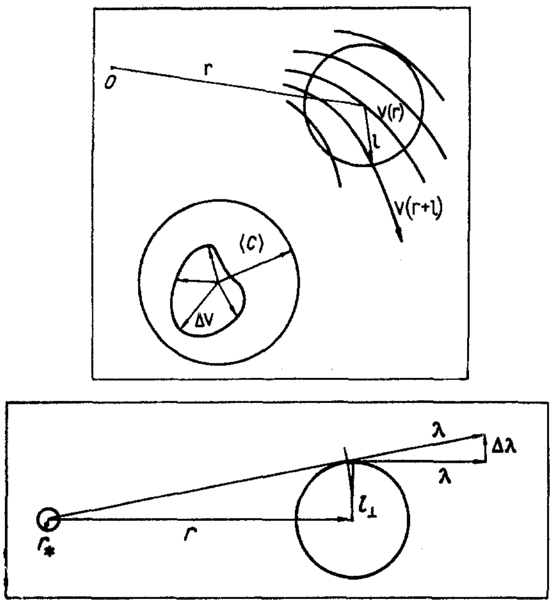

Fig. 1. Diagram showing the phenomena in the coordinate and velocity space (above) and in the case of a spherical source (below).

$$M_\infty = 0{,}9\mathrm{Kn}_0^{-0{,}4}. \tag{3}$$

Moreover, as in [5], it develops that in the cylindrical case none of the temperature components is frozen in.

A more detailed summary of the above-enumerated papers has recently been undertaken in [8].

2. Below we describe the proposed criterion (the $l-c$ criterion).

Let us examine an arbitrary steady-state expanding gas flow. Let the problem of the motion of this gas be solved in terms of a continuous medium and let us assume that we have determined the macroscopic velocity $\mathbf{V}(\mathbf{r})$, the density $\rho(\mathbf{r})$, and the temperature $T(\mathbf{r})$ at each point $\mathbf{r}$ of the space; consequently, for each point we know the mean free path $l(\mathbf{r})$ and the absolute value of the mean velocity of random motion $\langle c(\mathbf{r})\rangle$. We have to find the "boundary" Γ of the region for which the continuous-medium model is valid.

Each molecule in the gas flow participates in two motions: the transport motion with the macroscopic velocity $\mathbf{V}(\mathbf{r})$ of the continuous medium and the relative motion with the thermal velocity $\mathbf{c}(\mathbf{r})$. The former varies continuously from point to point, while the latter remains constant over the mean free path $l(\mathbf{r})$. Let us take some point $\mathbf{r}$ in the coordinate space (Fig. 1a) and surround it by a sphere of radius $l(\mathbf{r})$ (the l sphere). From the center of this sphere we draw the radius vector $\mathbf{l}(\mathbf{r}, \theta, \psi)$, where θ and ψ are the angles of the spherical coordinate system whose center is at $\mathbf{r}$. The macroscopic velocity at a chosen point on the surface of this sphere will be equal to $\mathbf{V}[\mathbf{r} + \mathbf{l}(\mathbf{r}, \theta, \psi)]$ and it will be a function of both θ and ψ. Let us turn to the velocity space and, in this space, let us construct a sphere with the radius $\langle c(\mathbf{r})\rangle$ (the c sphere). From its center we draw the vector

$$\Delta\mathbf{V}(\mathbf{r}, \theta, \psi) = \mathbf{V}(\mathbf{r}+\mathbf{l}) - \mathbf{V}(\mathbf{r}).$$

With a continuous change in ϑ and ψ the end of this vector in the velocity space will describe a certain ΔV surface. The proposed criterion involves the following: the continuous medium ceases to exist if the ΔV surface is tangential to the c sphere:

$$\sup|\Delta\mathbf{V}(\mathbf{r}_\Gamma, \theta, \psi)| = \langle c(\mathbf{r}_\Gamma)\rangle. \tag{4}$$

The set of points $\mathbf{r}_\Gamma$ forms the Γ surface.

If $l_\Gamma \ll \mathbf{r}_\Gamma$, the finite difference in the left-hand member of (4) can be expressed in terms of the derivative of the vector $\mathbf{V}$ with respect to the vector $\mathbf{l}$

$$\max|(\mathbf{l}, \nabla)\mathbf{V}| = \langle c(\mathbf{r}_\Gamma)\rangle.$$

To test how the proposed criterion "functions," let us take comparatively simple symmetrical gas flows from a sphere and cylinder. We will assume the gas to be nonviscous and nonheat-conducting, with a constant heat-capacity ratio. (It can be demonstrated that $r_\Gamma \lesssim r^0$, where r^0 is the radius at which viscous forces become evident [9].) In this case the flow parameters are expressed in terms of the familiar gasdynamic functions

$$r^{-\nu} = \left(\frac{\varkappa+1}{2}\right)^{\frac{1}{\varkappa-1}} \lambda \frac{\rho}{\rho_0}, \quad \frac{T}{T_0} = \left(\frac{\rho}{\rho_0}\right)^{\varkappa-1},$$
$$\frac{\rho}{\rho_0} = \left(1 - \frac{\varkappa-1}{\varkappa+1}\lambda^2\right)^{\frac{1}{\varkappa-1}} \tag{5}$$

($\nu = 1$ for a cylinder and $\nu = 2$ for a sphere), while expression (4) assumes the form ($l/l_0 = \rho_0/\rho$):

$$\max\left|\lambda\left[r_\Gamma + l_0\left(\frac{\rho_\Gamma}{\rho_0}\right)\right] - \lambda(r_\Gamma)\right| = \frac{\langle c_0\rangle}{a_0}\sqrt{\frac{\varkappa+1}{2}}\sqrt{\frac{T(r_\Gamma)}{T_0}}.$$

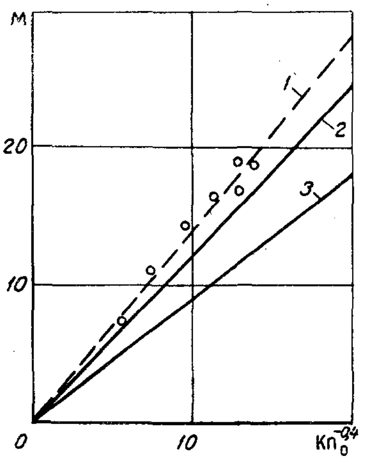

Fig. 2

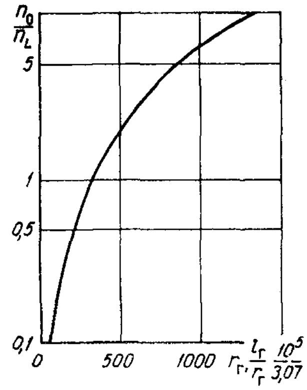

Fig. 3

Fig. 2. Comparison of the "frozen-in" Mach number as a function of the Knudsen number for a spherical source: 1) experiment [4], formula (1); 2) proposed criterion, formula (8'); 3) calculation with the model Boltzman equation [6], formula (3).

Fig. 3. Radius of "transition" and the corresponding mean free path as a function of the density within the spherical source.

To avoid complicating the matter with excessive computation, we will adopt the following assumptions: 1) the sought boundary of transition is sufficiently removed, i.e., $r_\Gamma \gg 1$; 2) at the boundary of transition the mean free path $l_\Gamma \ll r_\Gamma$ (in the following, these conditions will be verified). In this case

$$\lambda(r) \cong \sqrt{\frac{\varkappa+1}{\varkappa-1}}, \quad \left(\frac{\rho}{\rho_0}\right)^{-1} = r^{\nu}\left(\frac{\varkappa+1}{2}\right)^{\frac{1}{\varkappa-1}} \sqrt{\frac{\varkappa+1}{\varkappa-1}}. \tag{6}$$

Since $\lambda \cong \text{const}$, the maximum value for the difference $|\lambda(r_\Gamma + l_\Gamma) - \lambda(r_\Gamma)|$ can be achieved only in a direction approximately perpendicular to the streamline (Fig. 1b). From the similarity of the triangles constructed on r and λ, we have

$$\frac{\Delta\lambda}{\lambda} = \frac{l_\perp}{r_\Gamma}.$$

From (5) and (6) we obtain

$$r_\Gamma^{-\varkappa} = \frac{a_0}{\langle c_0 \rangle}\left(\frac{\varkappa+1}{2}\right)^{\frac{1}{\varkappa-1}} \left(\frac{\varkappa+1}{\varkappa-1}\right)^{\frac{3+\varkappa}{4}} l_0. \tag{7}$$

Let us compare this result with the transition radius of (2) for monatomic stable molecules ($\varkappa = 5/3$, $\beta = 1/2$). We have

$$r_1 = 0.225\,\mathrm{Kn}_0^{-3/5}, \tag{2'}$$

$$r_\Gamma = 0.333 l_0^{-3/5}. \tag{7'}$$

Considering that $\mathrm{Kn}_0 = l_0$ (since all linear dimensions are referred to r_*), we see that the shape of the function $r_\Gamma(\mathrm{Kn}_0)$ and the order of magnitude are properly described by the $l-c$ criterion.

Let us also carry out this comparison in terms of the function $M_\Gamma(\mathrm{Kn}_0)$. Since for $r \gg 1$ we have the relationship

$$M^2 \approx \left(\frac{\varkappa+1}{\varkappa-1}\right)^{\frac{\varkappa+1}{2}} r^{\nu(\varkappa-1)},$$

from (7) we find

$$M_\Gamma = \left(\frac{8}{\pi\varkappa}\right)^{\frac{\varkappa-1}{2\varkappa}} \left(\frac{\varkappa+1}{2}\right)^{-\frac{1}{\varkappa}} \left(\frac{\varkappa+1}{\varkappa-1}\right)^{\frac{3-\varkappa}{4\varkappa}} l_0^{-\frac{\varkappa-1}{\varkappa}}. \tag{8}$$

For $\varkappa = 5/3$ we have

$$M_\Gamma = 1.21 Kn_0^{-0.4}, \tag{8'}$$

which is a result that is in very good agreement with expressions (1) and (3). For clarity, each of the three functions has been plotted in Fig. 2.

Thus comparing the conclusion drawn with the aid of the proposed criterion with the results of the solutions for the Boltzmann equation and the experimental results for spherical flow demonstrates their satisfactory agreement, despite the intrinsic contradiction of the criterion (the existence of a continuous medium at the point r_Γ is initially assumed, and then rejected). This enables us to believe that the l–c criterion of (4) is applicable to the general case, since it is free of any assumptions with regard to the geometry of the flow.

Figure 3 shows r_Γ and l_Γ/r_Γ as a function of n_0/n_L for a gas of monatomic stable molecules. The conditions $r_\Gamma \gg 1$ and $l_\Gamma/r_\Gamma \ll 1$ (assumed only to simplify the calculations and to have no bearing on the essential nature of the l–c criterion) have been satisfied here.

For the case of a cylindrical source, expressions similar to (7) and (8) are of the form

$$r_\Gamma^{-\frac{\varkappa-1}{2}} = \sqrt{\frac{\pi\varkappa}{8}}\left(\frac{\varkappa+1}{2}\right)^{\frac{1}{\varkappa-1}}\left(\frac{\varkappa+1}{\varkappa-1}\right)^{\frac{\varkappa+3}{4}} l_0, \quad M_\Gamma^{-1} = \sqrt{\frac{\pi\varkappa}{8}}\left(\frac{\varkappa+1}{2}\right)^{\frac{1}{\varkappa-1}}\sqrt{\frac{\varkappa+1}{\varkappa-1}}\, l_0.$$

The l–c criterion thus indicates the existence of a transition "surface," which is in agreement with [3] and contradicts the conclusions of [5] and [6]. However, since all of the above-cited references are based on the approximate BGK model of the Boltzmann equation, the question of the existence or absence of such a "surface" in the cylindrical case remains open, a fact which is enhanced by the conditional nature of the concept itself.

The author expresses his gratitude to V. N. Zhigulev, F. A. Kukanov, and E. A. Romishevskii for their useful discussions.

NOTATION

V, **c**	are, respectively, the macroscopic and thermal velocities of the gas molecules;
$\langle c \rangle$, L	are, respectively, the average thermal velocity and mean free path;
ρ, n, T	are, respectively, the mass and numerical densities and the absolute temperature of the gas;
r	is the radius vector;
$\lambda = V/a_*$, $M = V/a$	are the reduced velocity and the Mach number;
a	is the speed of sound;
$\varkappa$	is the ratio of heat capacities;
β	is the exponent in the function of molecular interaction (for stable molecules $\beta = 1/2$, and for Maxwellian molecules $\beta = 0$);
Kn	is the Knudsen number;
$n_L = 2.69 \cdot 10^{25}\ m^{-3}$	is the Lohschmidt number.

Subscripts

0	denotes the stagnation conditions;
*	denotes the parameters on the sonic line;
Γ	denotes the parameters at the "transition surface."

All linear dimensions are referred to the source radius r_*, $l = L/r_*$.

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